Fun with Formulas !

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Introduction

Formulas can be fun. They often can be made to look simple, transparent and thus beautiful!
This can be achieved with some simple rules and a few tricks of the trade.
This paper is a sample of some simple formulas, collected during my career as a physicist.

Some Examples:

- the relativistic formulas of Einstein
- the magic triangle formed by the derivatives of the corresponding relativistic parameters
- the use of binomial curves to approximate a variety of functions, like beam profiles, the fringe field of magnets, the flux and brightness of synchrotron radiation etc.
- the simple representation of phase space ellipses used in beam transport calculations

The following material was presented (but not published) at a seminar talk given at the CERN Accelerator School on Synchrotron Radiation and Free Electron Lasers, Brunnen, Switzerland, 2-9 July 2003

$$e^{i\pi} = -1$$

This formula from Euler combines beautifully 3 fundamental numbers in mathematics

another „gem“ from Euler is:

$$3^3 + 4^3 + 5^3 = 6^3$$

in the the same spirit one can write:

$$1^3 + 2^3 + 3^3 + \dots n^3 = (1 + 2 + 3 + \dots n)^2$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{n^2 (n-1)^2}{4} \qquad\qquad [\frac{n(n-1)}{2}]^2$$

(I figured this out myself, but I am sure it exists somewhere in the literature, but I could not find yet the proper reference)

philosophy for formulas

- simplify formulas
- formula should indicate the proper dimensions
- use units of 1'000 (cm should not exist in formulas!)
- choose right scales for plots

example:

$c = 3 \cdot 10^8$ m/s ?? better is:

$c = 0.3 \cdot 10^9$ m/s or 300 m/μs or 0.3 mm/ps !!

$\mu_0 = 4\pi\ 10^{-7}$ Vs/Am ?? better is:

$\mu_0 = 0.4\pi$ μH/m = 0.4π T/(kA/mm) !!

how to avoid akward numbers in electrodynamics

electron mass: $m = 9.11 \cdot 10^{-31}$ kg (?) => forget it !

use $mc^2 \equiv eU_o$, $U_o = .511$ MV

$\varepsilon_o = 8.854 \cdot 10^{-12}$ As/Vm (?) => forget it !

use $\mu_o \varepsilon_o = 1/c^2$ with $\mu_o = 0.4\ \pi\ \mu$H/m

introduce impedance Z_o :

$$Z_o \equiv \frac{1}{4\pi\varepsilon_0 c} = \frac{\mu_0}{4\pi} c = 30\ \Omega$$

($\equiv 29.9792458\ \Omega$)

Alfven current I_A (used in space charge calculations):

$I_A = 4\pi\varepsilon_o mc^3/e$ (?) => forget it ! (similarly for "perveance")

use instead "Ohms-law" : $I_A = U_o/Z_o = 511\ \text{kV}/30\Omega = 17$ kA

charged particle in a magnetic field B_o

=> Larmor-frequency

$$\omega_o = \frac{e B_0}{m \gamma}$$

electron: $\dfrac{e}{m} = 1.76 \cdot 10^{11}$ C/kg (?) => forget it !

use $\dfrac{e}{2\pi m} = \dfrac{c^2}{2\pi U_0} =$ 28 GHz/T (15.25 MHz/T for protons)

use of magic numbers
to memorize some relations, e.g.

photon energy ⇔ wavelength

$\varepsilon\,\lambda$ = 1240 eV nm (=hc)

trick => take square root !

VUV-region	35 eV ⇔ 35 nm
soft X-rays	1.1 keV ⇔ 1.1 nm
„old fashioned“	3.5 keV ⇔ 3.5 Å

„infrared-people“ use wavenumber k in [cm^{-1}]

100 cm^{-1} ⇔ 100 μm

(correlations for arbitrary numbers are then quickly estimated by multiplication resp. division)

"nice" numbers in the SLS facility:

increase of energy during design studies:

E :	1.8 => 2.1 => 2.4 => 2.7 GeV
$B\rho$:	6 => 7 => 8 => 9 Tm

circumference of storage ring:

$L = 3 \cdot 96\ \text{m} = 288\ \text{m}$

$\lambda_{rf} = 0.6\ \text{m}$

=> harmonic = $480 = 2^5 \cdot 3 \cdot 5$

TBA-dipoles 8^0 , 14^0 , 8^0

bending angles = 10^0 /m

$\rho = 18\text{m}/\pi = 5.73\ \text{m}$

circumference of booster:

$L = 3 \cdot 90\ \text{m} = 270\ \text{m}$

=> harmonic = $450 = 2 \cdot 3^2 \cdot 5^2$

Einstein triangle

in relativistic equations use dimensionless quantities for

velocity, energy and momentum on a democratic basis!

=> Einstein triangle and magic triangles for logarithmic derivatives

velocity: $v = \beta c$

total energy: $E = \gamma E_0$, $E_0 = mc^2 = eU_0$ (0.511 MeV for electron)

$\gamma = (1-\beta^2)^{-1/2}$,

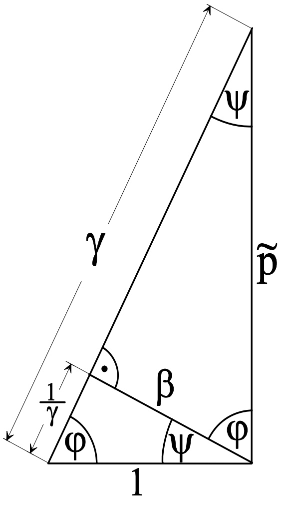
momentum: $p = \tilde{p}\, E_0/c = \tilde{p}\, mc$

$$\tilde{p} = \beta\gamma$$

"Pythagoras"- connections give Einstein triangle and derivatives

$$1 + \tilde{p}^2 = \gamma^2 \qquad \tilde{p}\,d\tilde{p} = \gamma\, d\gamma$$

$$\beta^2 + \frac{1}{\gamma^2} = 1 \qquad d\gamma = \beta\gamma^3 d\beta$$



for relativistic formula :
use trigonometric formulas!

$$\gamma = \frac{1}{\sqrt{(1-\beta^2)}}$$

$$if \quad \beta = \sin\varphi, \quad then \quad \gamma = \frac{1}{\cos\varphi}$$

$$\beta\gamma = \tan\varphi$$

the following table gives some easy reference values for

quick estimates or for calculations with a **pocket calculator**!

φ	**β = sinφ**	cosφ	**γ= 1/cosφ**	**p = βγ = tanφ**
30^0	0.5	0.87	1.15	0.58
36.9^0	**0.6**	**0.8**	1.25	0.75
45^0	0.71	0.71	1.41	1
53.1^0	**0.8**	**0.6**	1.67	1.33
60^0	0.87	0.5	2	1.71

highly relativistic case

$\gamma >> 1$, $\beta \approx 1$ => use angle ψ

$\beta \equiv \cos\psi \approx 1 - \psi^2 / 2$

$1/\gamma \equiv \sin\psi \approx \psi$

$\tilde{p} \equiv \beta\gamma = 1/\tan\psi \approx 1/\psi$

$$1 - \beta \approx \frac{1}{2\gamma^2}$$

a) race to the moon between electron and photon:

electron "looses" by

$\delta L = (1 - \beta)\, L \approx \frac{1}{2\gamma^2} L$

$$\delta L \approx \frac{50m}{E^2[GeV]}$$

SLS:	2.4 GeV,	δL = 8 m
ESRF:	6 GeV,	δL = 1.4 m
LEP II:	100 GeV,	δL = 5 mm

b) race over one undulator period λ_u : if electron is just one wavelength λ behind photon (slippage) => positive interference

$$\delta L = \lambda = \frac{1}{2\gamma^2} \lambda_u \,(1 + K^2 / 2)$$

$K = 0.0934\, B[T]\, \lambda_u\, [mm]$

↑

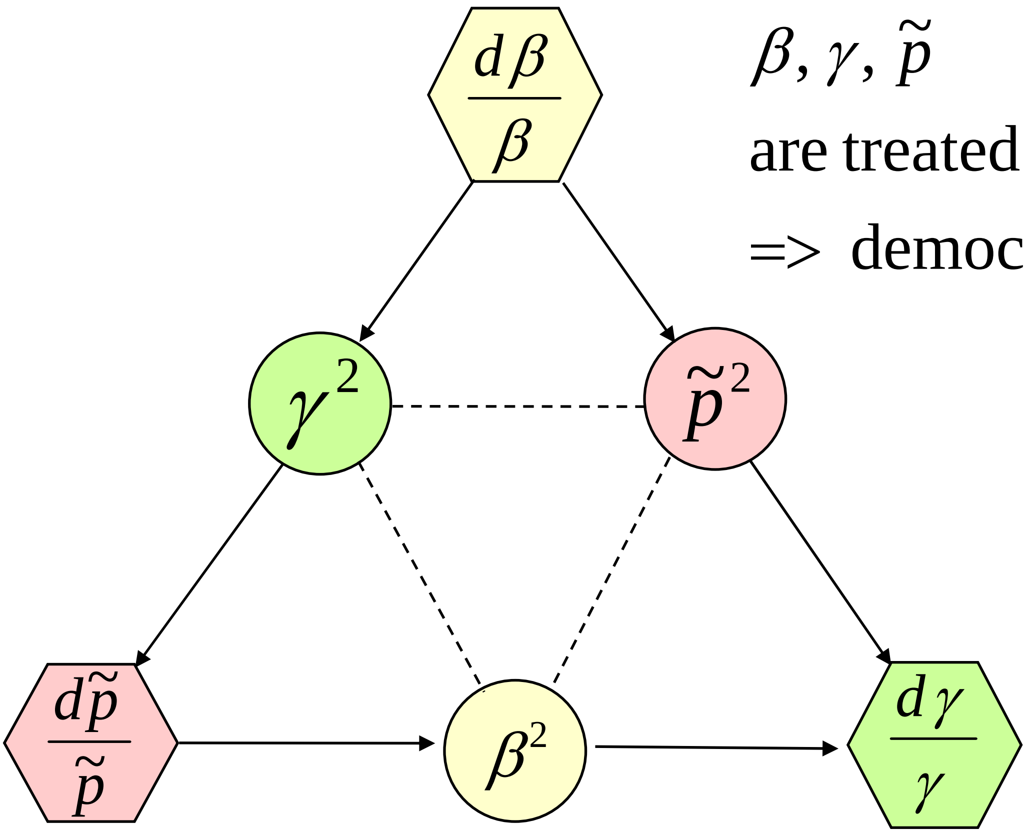
detour due to slalom in B-field !

magic triangle
with logarithmic derivatives of
relativistic parameter $\beta, \gamma, \tilde{p} \equiv \beta\gamma$

$$\frac{d\tilde{p}}{\tilde{p}} = \frac{d\beta}{\beta} + \frac{d\gamma}{\gamma}$$

$$\frac{d\tilde{p}}{\tilde{p}} = \gamma^2 \frac{d\beta}{\beta}$$

$$\frac{d\gamma}{\gamma} = \tilde{p}^2 \frac{d\beta}{\beta}, \qquad \frac{d\gamma}{\gamma} = \beta^2 \frac{d\tilde{p}}{\tilde{p}}$$



$\beta, \gamma, \tilde{p}$

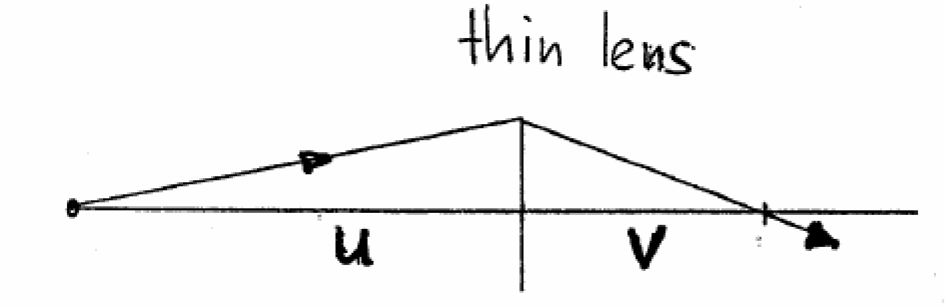
are treated equally

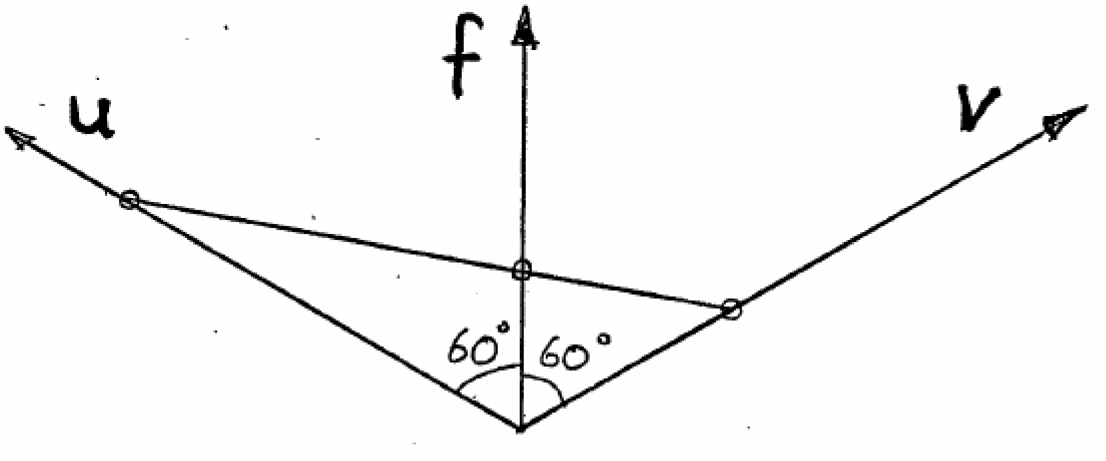
=> democracy!

W.Joho

the lens equation
solved graphically for a
thin lens with focal length f

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$





same graph for resistances in parallel,

capacitances in series etc.!

AG-focusing

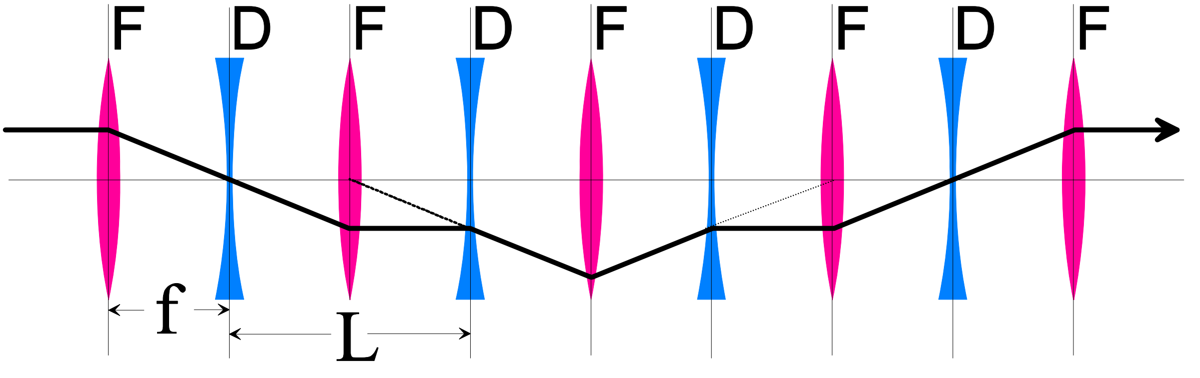
simple example of alternative gradient focusing:

$\Rightarrow$ FODO-lattice with thin lenses (focal length f)

if L = 2f => construction is possible by hand !

it takes 6 periods to get a 360^0-oscillation

i.e. the phase advance/period is $\psi = 60^0$



exact solution with transfer matrices gives

$$\sin\frac{\psi}{2} = \frac{L}{4f}$$

for L = 2f => $\psi = 60^0$

for L = 4f => $\psi = 180^0$ (instability !)

general binomial curves

$F(x) = F_{max}\, y(u)$, $u \equiv x/x_L$, $y(0) = 1$

3 general cases :

short range : $s > 0$

$$y = (1 - u^n)^{1/s} \quad , \quad (0 \le u \le 1)$$

inverse : $$u = (1 - y^s)^{1/n} \quad , \quad (0 \le y \le 1)$$

long range : $s < 0$

$$y = (1 + u^n)^{1/s} \quad , \quad (0 \le u \le \infty)$$

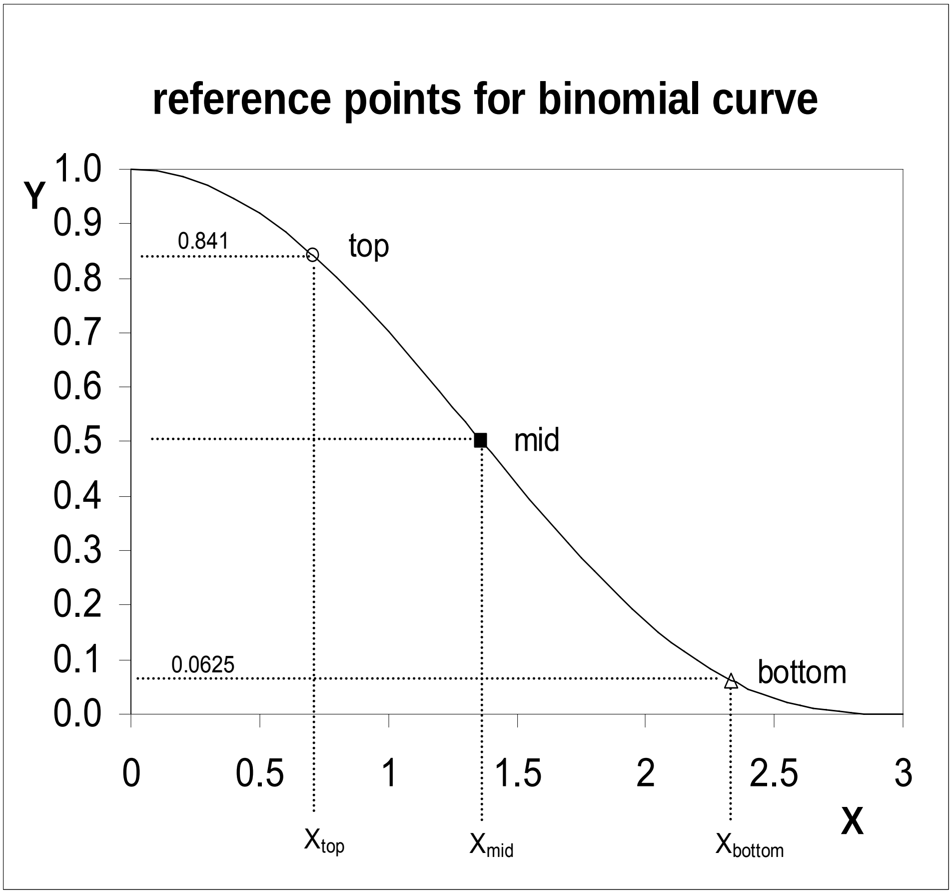
inverse : $$u = (y^s - 1)^{1/n} \quad , \quad (0 \le y \le 1)$$

exponentials : (limit $s = 0$)

$$y = \exp(-u^n) \quad , \quad (0 \le u \le \infty)$$

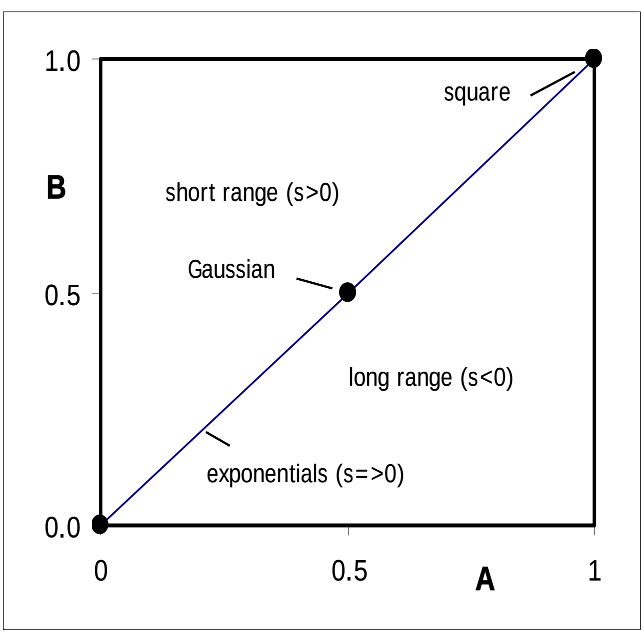
inverse : $$u = (-\ln y)^{1/n} \quad , \quad (0 \le y \le 1)$$

Classification of binomial curves



$$binomial: \quad y(x) = [1 - sign(s)(\frac{x}{x_L})^N]^{\frac{1}{S}}$$

any 3 reference points will give a fit for N, S and x_L. But the chosen points top, mid and bottom allow a convenient classification in the (A,B)-Diagram



Classification of binomials in (A,B) - Diagram

$$y^{top} = 0.5^{1/4} = 0.841$$

$$y^{mid} = 0.5$$

$$y^{bottom} = 0.5^4 = 0.0625$$

$$A \equiv \frac{x_{top}}{x_{mid}}, \quad B \equiv \frac{x_{mid}}{x_{bottom}}$$

properties of binomials

1. $y(u) = (1 \pm u^n)^{1/s}$ is monotonically decreasing, but transformations like $F(x) = A\, x^\alpha\, y(u)$ allow representations of functions which are not monotonic (e.g. Flux- or Brightness- curves)

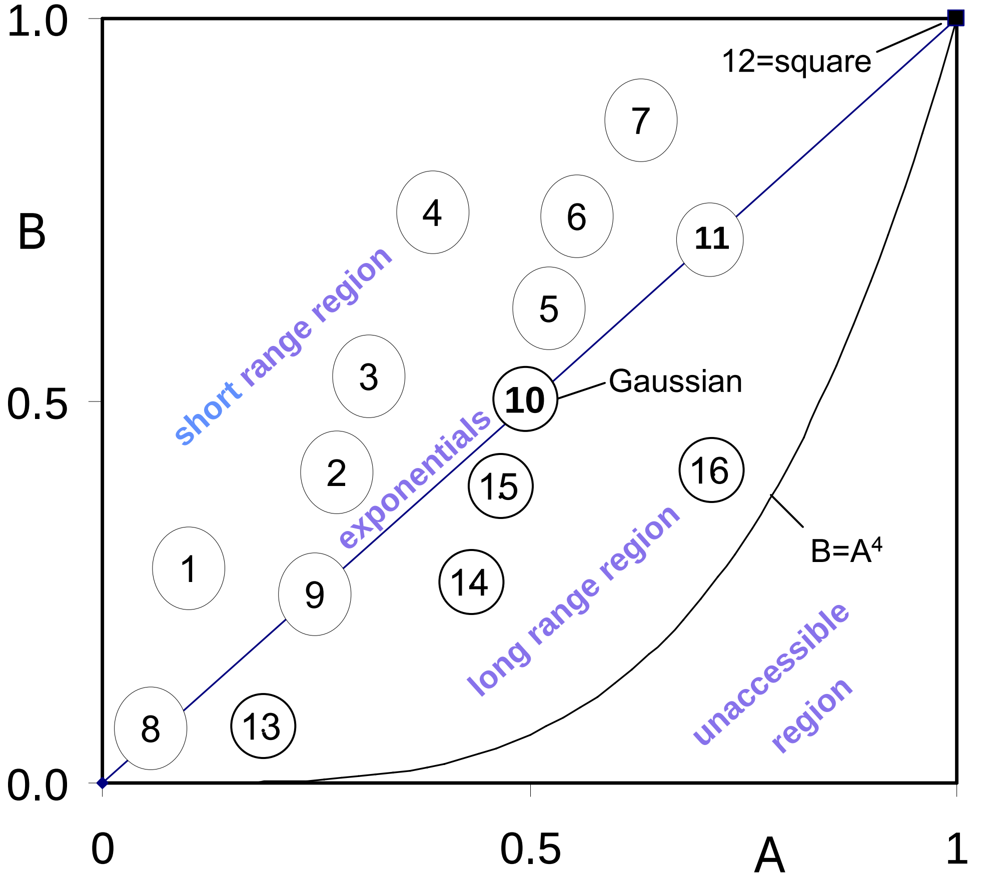
2. inverse function exists : $y \Leftrightarrow u$, $n \Leftrightarrow s$

3. 4 free parameter:
    1. F_{max} gives scaling in y
    2. x_L gives scaling in x , (u=1)
    3. n determines flatness at x = 0
    4. s determines tail at large x

4. Fit to data using the 4 parameter with programs like MATLAB", "IGOR" or "Excel" (insert , name , define) + (extras , solver)

5. (A , B) – diagram allows rough estimates of parameter n and s

typical profiles y(u) in (A,B)-plot



1 $y = 1 - \sqrt{u}$

2 $y = (1-u)^2$ parabola, convex

3 $y = 1 - u$ triangle

4 $y = \sqrt{1-u}$

5 $y = (1-u^2)^2$ biquadratic

6 $y = 1 - u^2$ parabola, concave

7 $y = \sqrt{1-u^2}$ quarter circle

8 $y = e^{-\sqrt{u}}$

9 $y = e^{-u}$ exponential decay

10 $y = e^{-u^2}$ Gaussian

11 $y = e^{-u^4}$

12 $y = 1$ square

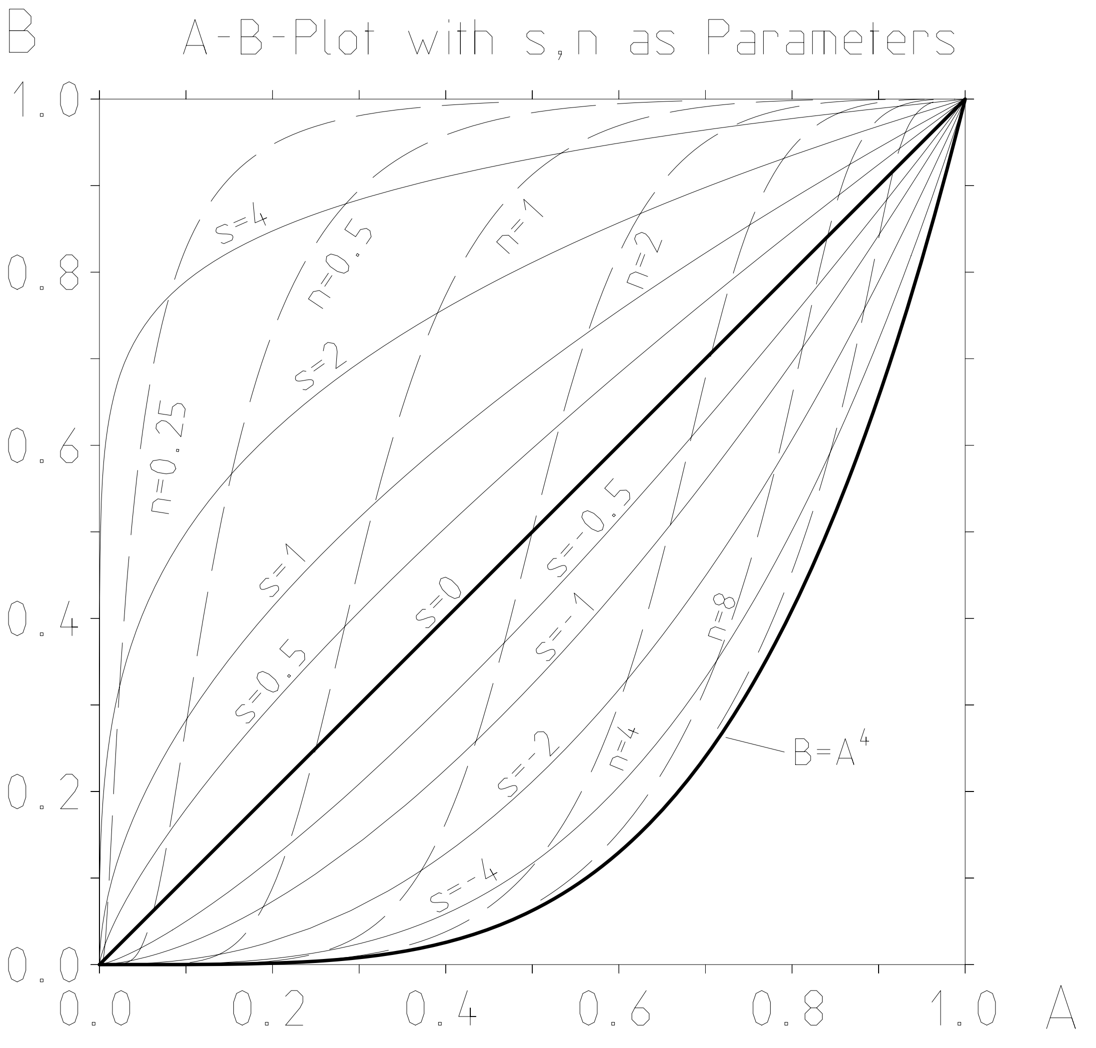
13 $y = \dfrac{1}{1+u}$

14 $y = \dfrac{1}{1+u^2}$ Lorentzian

15 $y = \dfrac{1}{(1+u^2)^2}$ bi-Lorentzian

16 $y = \dfrac{1}{(1+u^7)^{1/3}}$ magnetic fringe field

Classification of „Profiles“ y(u) with binomial curve



normalization: y(0)=1, y(∞)=0

determine 3 reference points of y(u):

top point : $y_T= 0.5^{1/4} =0.861$,

mid point: $y_M=0.5$,

bottom point: $y_B= 0.5^4 =0.0625$

build ratios **$A \equiv u_M/u_T$, $B \equiv u_B/u_M$**

get estimate for s and n from diagram

s>0: short range; $y = (1-u^n)^{1/s}$, u < 1, A<B

s=0: exponentials; $y = \exp(-u^n)$, A=B, Gaussian: n=2, A=B=0.5

s<0: long range; $y = (1+u^n)^{1/s}$, A>B

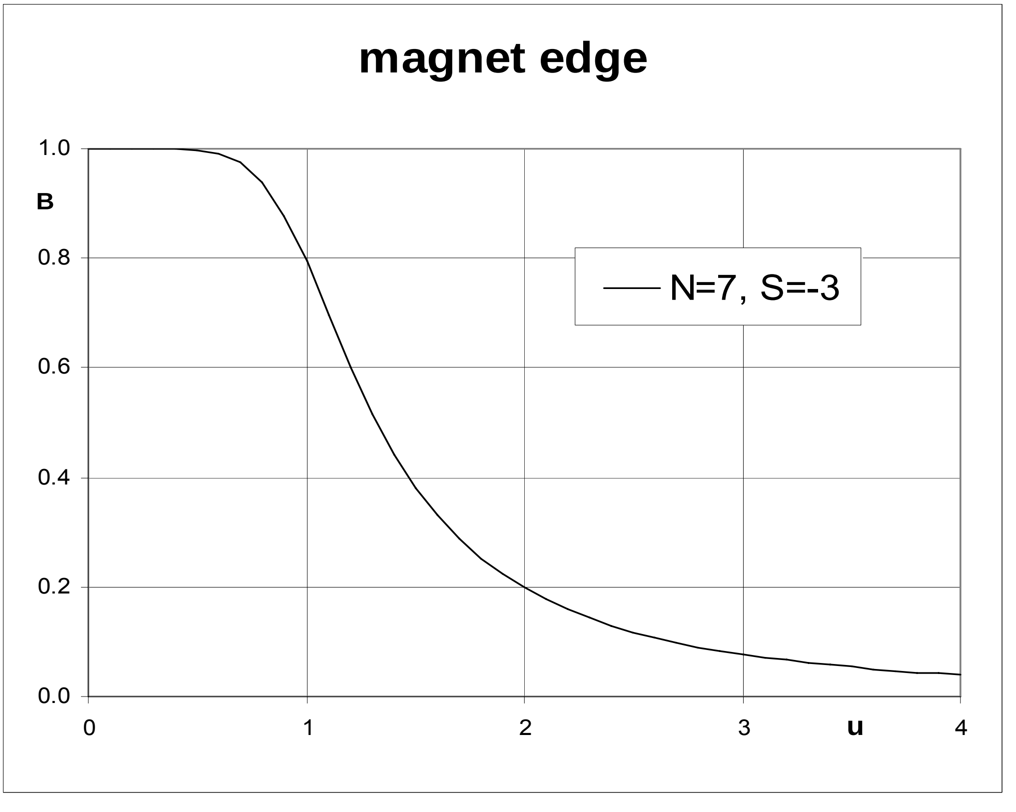
magnetic fringe field with binomial

$$B(x) \approx \frac{1}{(1+u^7)^{1/3}}, \quad u \approx \frac{x - x_0}{gap}$$

$$x_0 \approx x(80\%\ field) - gap$$

$$inverse\ :$$

$$u = \left(\frac{1}{B^3} - 1\right)^{1/7}$$



Flux Spectrum of synchrotron radiation

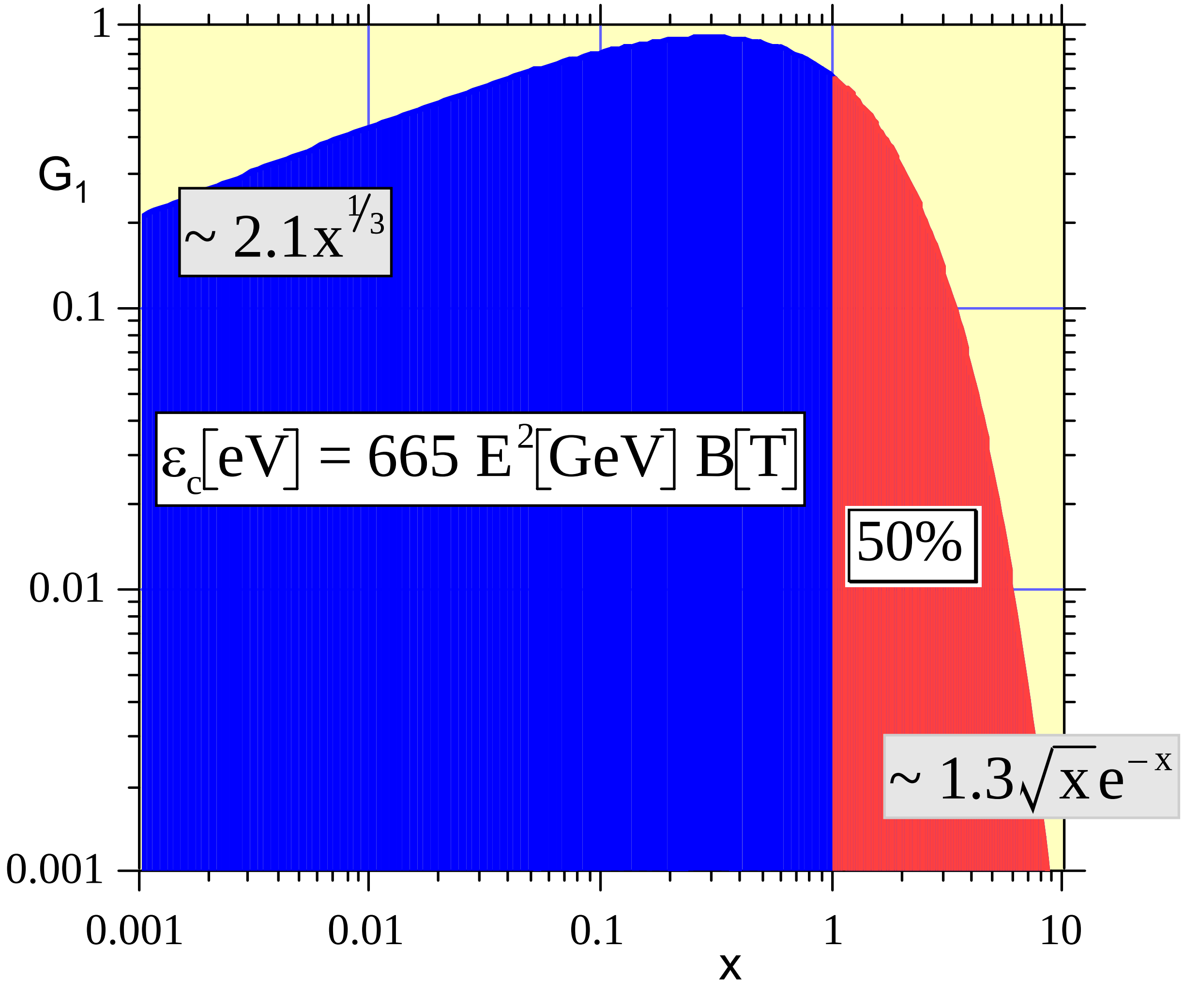
spectral flux F of electrons with energy E and current I from a bending magnet with magnetic field B.

$F = 2.46 \cdot 10^{13}\ E[\mathrm{GeV}]\ I[\mathrm{A}]\ G_1(x)$

(photons/(s ·mrad ·0.1% bandwidth)

$x=\varepsilon/\varepsilon_c$, ε = photon energy , ε_c = critical photon energy

$$G_1(x) = x\int_x^\infty K_{5/3}(x')dx'$$



Flux-Spectrum of Synchrotron Radiation from Bending Magnet with Field B

$$G_1(x) = x\int_x^{\infty} K_{5/3}(x')\,dx'$$

$$x \equiv \frac{\varepsilon}{\varepsilon_c}, \quad \varepsilon = photon\ energy$$

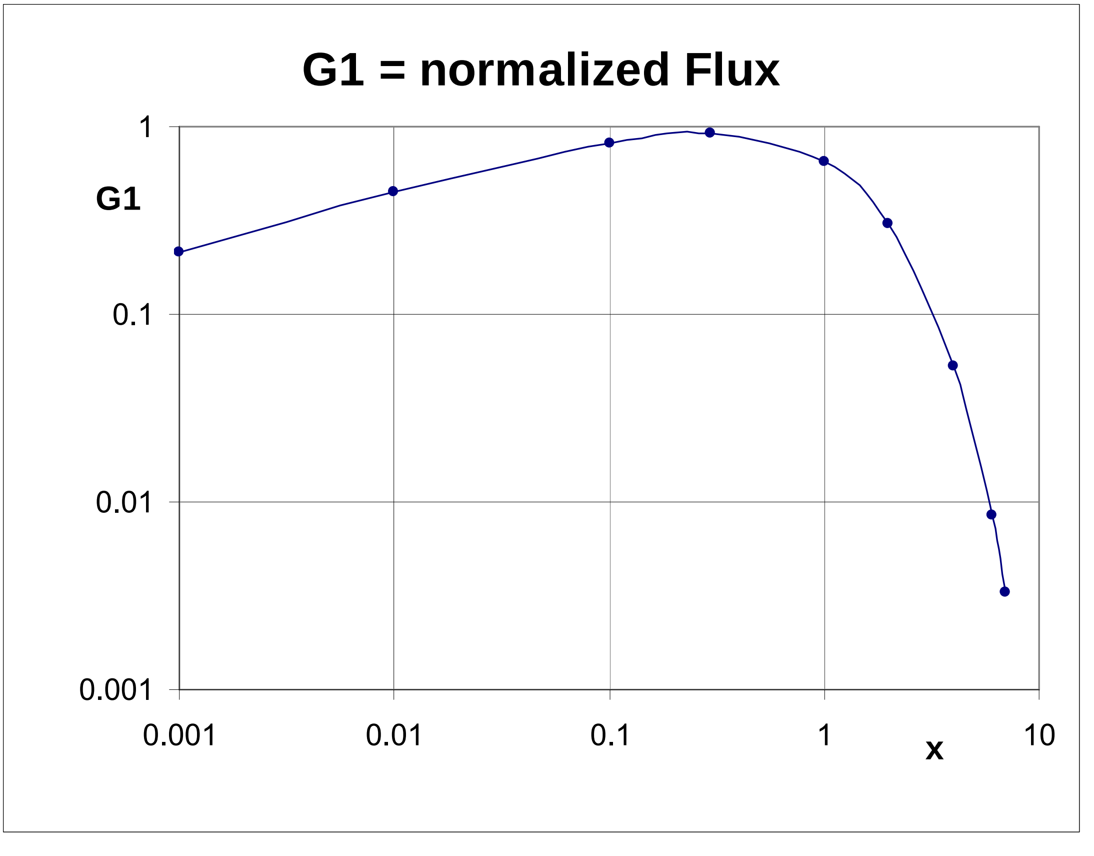
$$\varepsilon_c = 665\,eV \cdot E^2[GeV] \cdot B[T]$$

Fit with $G_1(x) = A\,x^{1/3}\,g(x)$,

$$g(x) = [(1-(\frac{x}{x_L})^N]^{\frac{1}{S}}$$

fit with 8 data points to ±1.5%:

$A = 2.11$, $N = 0.848$, $x_L = 28.17$, $S = 0.0513$



W.Joho

Brightness of Synchrotron Radiation from Bending Magnet with Field B

$$H_2(x) \equiv x^2 K_{2/3}^2(\frac{x}{2}) \,,$$

$$x \equiv \frac{\varepsilon}{\varepsilon_c}, \;\; \varepsilon = photon\ energy$$

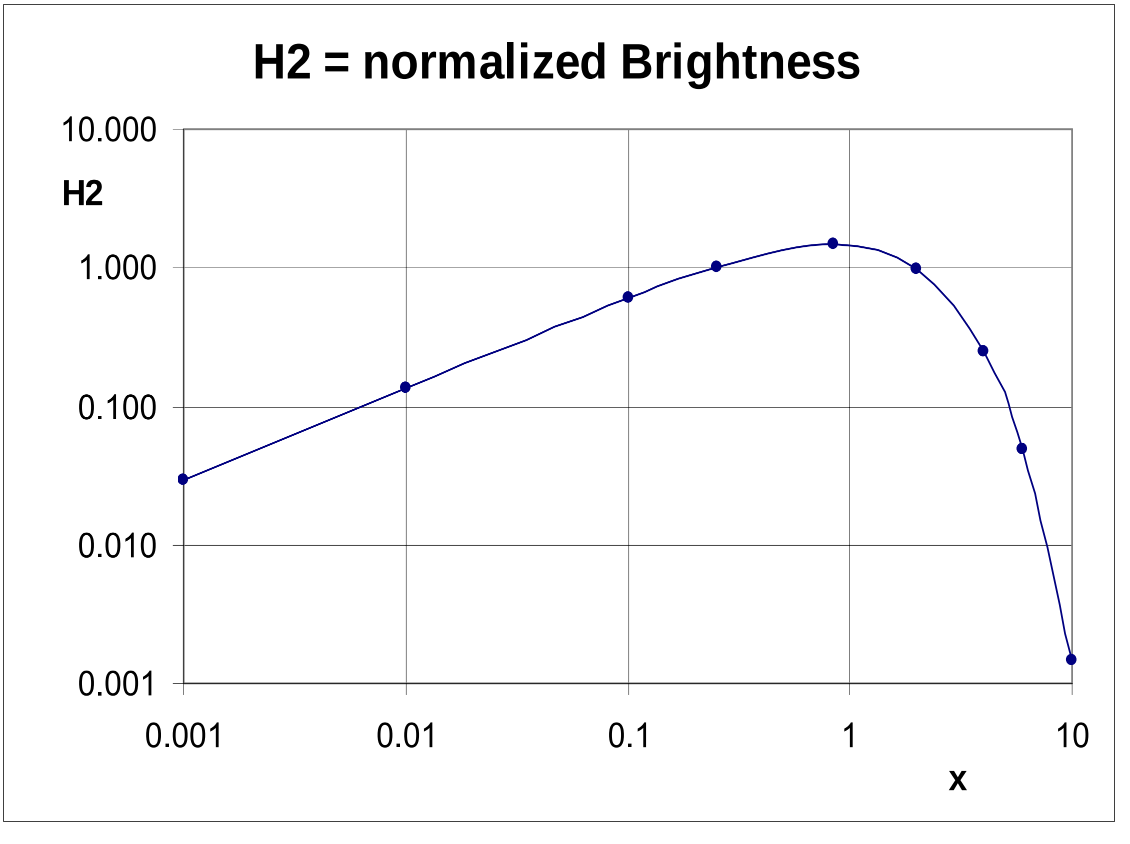
$$\varepsilon_c = 665 eV \cdot E^2[GeV] \cdot B[T]$$

Fit with $H_2(x) = A\, x^{2/3}\, h(x)$,

$$h(x) = \exp[-(\frac{x}{x_L})^N]$$

fit with 8 data points to ±2%:

A = 2.95 , N = 1.11 , x_L = 1.336



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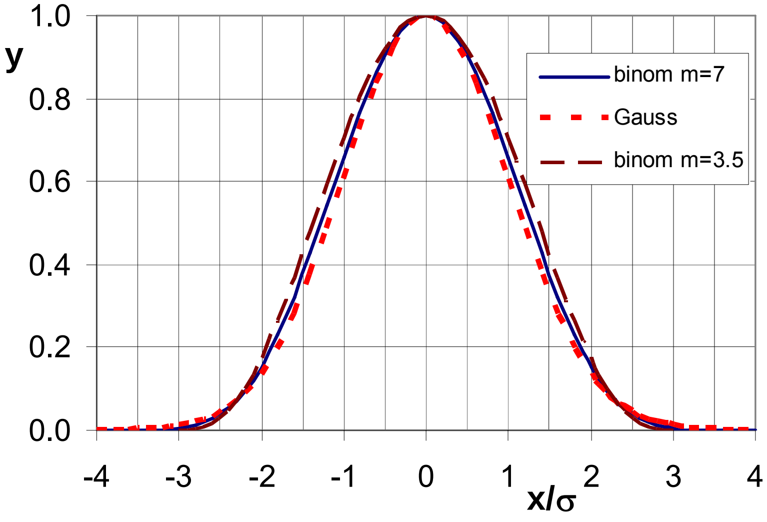
representations of beam profiles with binomials

1) Gaussian: $y = e^{-1/2(x/\sigma)^2}$

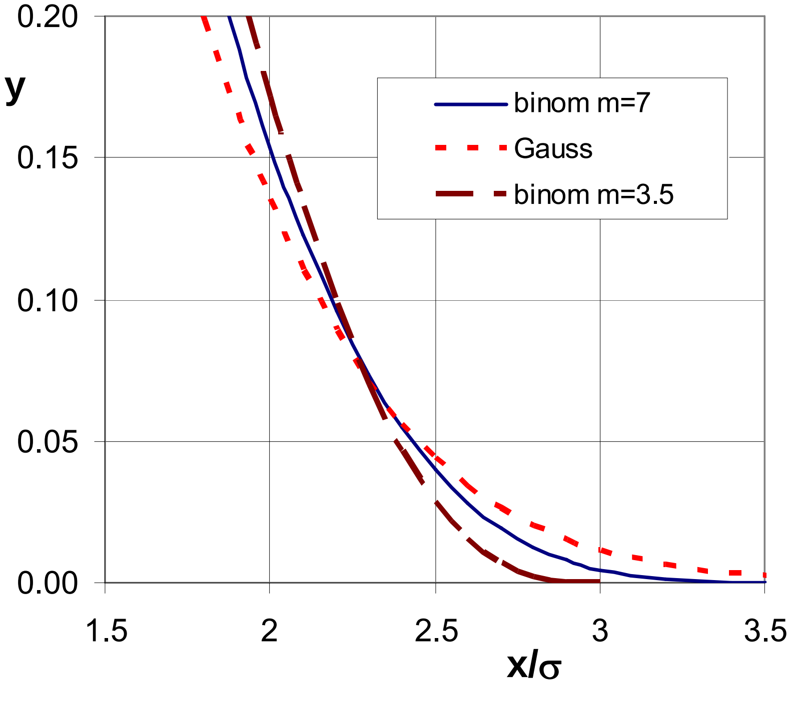
2) Binomials :

$$y = [(1-(\frac{x}{x_L})^2]^{m-1/2}, \quad x \le x_L , \quad x_L = \sqrt{2(m+1)}\,\sigma$$

clipped tails at x_L (e.g. m=3.5 , x_L= 3σ , m=7 , x_L= 4σ)



Profiles



Tails of Profiles

(full width at 10% level ≈ 4.4 σ for large range of m)

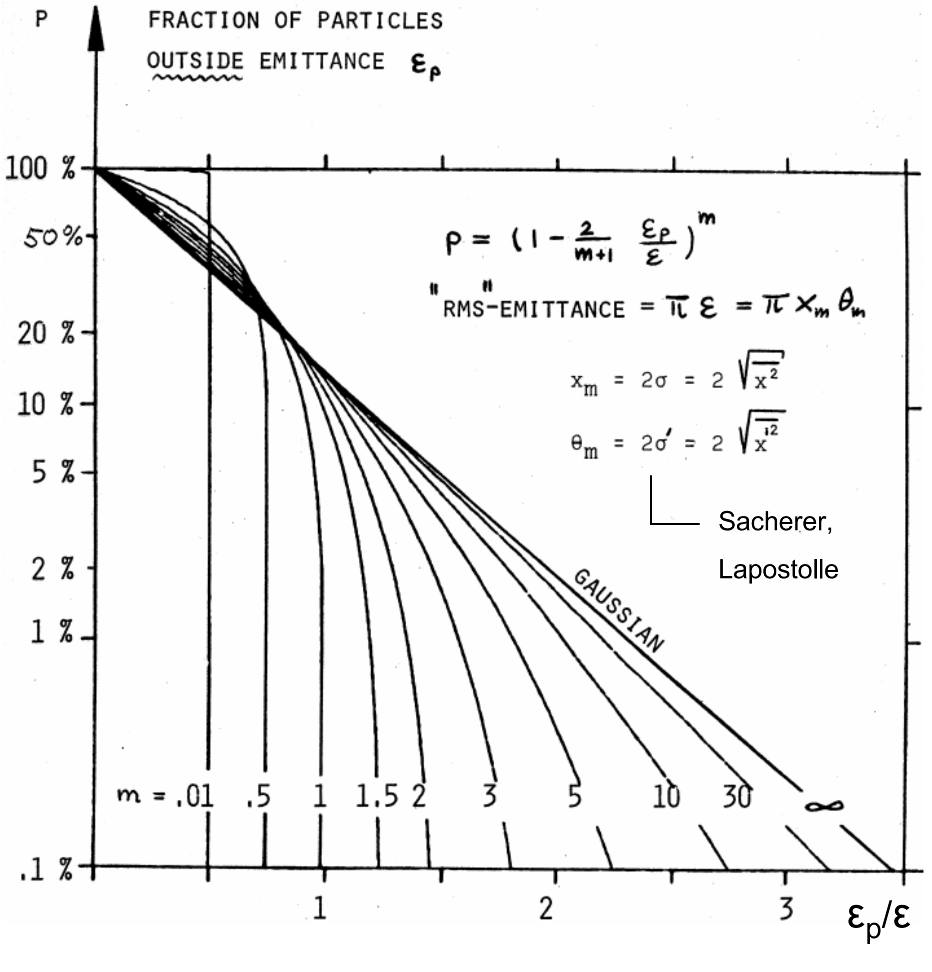
clipped binomial phase space densities

$$\rho(x,x') = (1-a^2)^{m-1}$$

$$(a^2 \equiv u^2 + v^2 \le 1\,, \quad u \equiv \frac{x}{x_L}, \quad v \equiv \frac{x'}{x'_L})$$

$$\text{projected profile}: \; y(x) = (1-u^2)^{m-1/2}$$

we get again a binomial with the exponent reduced by 1/2



for m≥1.5 the curves have a crossing point at $\varepsilon_p \approx \varepsilon$ and p ≈ 13%; i.e. ca. 87% of all particles are inside an ellipse with emittance ε=(2σ)·(2σ‘) , independent of m.

For a Gaussian distribution we have $p=\exp(-2\varepsilon_p/\varepsilon)$, which gives a straight line in this diagram (m=∞).

W.Joho, 1980
PSI report TM 11-4

Representation of rms beam ellipse in phase space (x, x')

For an arbitrary distribution of particles define the statistical values:

=> shift first origin such that $\langle x \rangle = \langle x' \rangle = 0$

$$\langle x^2 \rangle \equiv \sigma^2 \quad , \quad \langle x'^2 \rangle \equiv \sigma'^2 \quad , \quad \langle x\, x' \rangle$$

the rms-emittance ε is then defined as:

$$\varepsilon^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x\, x' \rangle^2$$

the traditional representation of the rms-ellipse is given with the Courant-Snyder invariant :

$$\gamma\, x^2 + 2\alpha\, x\, x' + \beta\, x'^2 = \varepsilon$$

the parameter α, β, γ are defined by:

$$\beta = \sigma^2/\varepsilon \quad , \quad \gamma = \sigma'^2/\varepsilon \quad , \quad \alpha = - \langle x\, x' \rangle/\varepsilon$$

with the relation $$\gamma\, \beta = 1 + \alpha^2$$

the graph representing this ellipse is akward to remember

and not easy to plot!

The parametric representation of the rms beam ellipse in phase space (x, x')

$x = \sigma \cos \varphi$

$x' = \sigma' \sin(\varphi + \chi)$

($0 \leq \varphi \leq 2\pi$ = "running parameter")

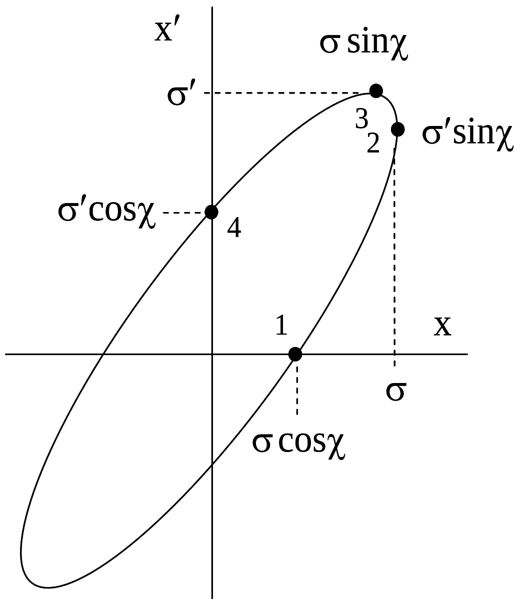
but what is phase shift χ ?

$\sin \chi \equiv r_{12} \equiv \frac{<xx'>}{\sigma\sigma'}$ = correlation parameter

(in a drift, χ is just the phase advance ψ from the waist position)

this representation of the rms-ellipse is easy to remember and easy to plot!

(χ = 0 gives a circle)



Some useful quantities are easy to guess from the factors

sinχ or cosχ (χ is 0 at a waist!) and dimensional arguments

(using m, mm and mrad)

emittance:	$\varepsilon = \sigma\sigma' \cos\chi$	[mm mrad]
slope of envellope:	$d\sigma/ds = \sigma' \sin\chi$	[mrad]
virtual waist size:	$x_w = \sigma \cos\chi$	[mm]
β-function at virtual waist:	$\beta_{min} = (\sigma/\sigma') \cos\chi$	[m]
distance from virtual waist:	$L_w = (\sigma/\sigma') \sin\chi$	[m]
	$= \beta_{min} \tan\chi$	
phase advance from virtual waist:	$\psi = \chi$	

the dictionary between the 2 representations is:

$$\alpha = -\tan\chi = -\frac{<xx'>}{\varepsilon} \quad (= -x_3/x_1 = -x_2'/x_4') \quad [1]$$

$$\beta = \frac{\sigma^2}{\varepsilon} = \frac{\sigma}{\sigma'\cos\chi} \quad (= x_2/x_4') \quad [m]$$

$$\gamma = \frac{\sigma'^2}{\varepsilon} = \frac{\sigma'}{\sigma\cos\chi} \quad (= x_3'/x_1) \quad [m^{-1}]$$

(as a check : $\beta\gamma = 1/\cos^2\chi = 1 + \tan^2\chi = 1 + \alpha^2$)

Example: convolution of the electron beam ellipse (x_1, x_1'), with parameter $\sigma_1, \sigma_1', \chi_1$ and the diffraction limited photon beam (x_2, x_2'), with parameter $\sigma_2, \sigma_2', \chi_2$ from an undulator.

Simple recipe:

add variances and correlations linearly

to form the combined ellipse (X,X') with parameter Σ, Σ', χ

$$\langle X^2\rangle = \langle x_1^2\rangle + \langle x_2^2\rangle$$

$$\langle X'^2\rangle = \langle x_1'^2\rangle + \langle x_2'^2\rangle$$

$$\langle XX'\rangle = \langle x_1 x_1'\rangle + \langle x_2 x_2'\rangle$$

or

$$\Sigma^2 = \sigma_1^2 + \sigma_2^2$$

$$\Sigma'^2 = \sigma_1'^2 + \sigma_2'^2$$

$$\Sigma\,\Sigma' \sin\chi = \sigma_1\sigma_1' \sin\chi_1 + \sigma_2\sigma_2' \sin\chi_2$$

the convoluted emittance is

$$\varepsilon = \Sigma\Sigma' \cos\chi \quad (\varepsilon \geq \varepsilon_1 + \varepsilon_2)$$

with the dictionary one can, if necessary, transform these values back to the Courand-Snyder values α, β, γ .

correlations x ⇔ y

example:

income and research for 50 US companies in 1976

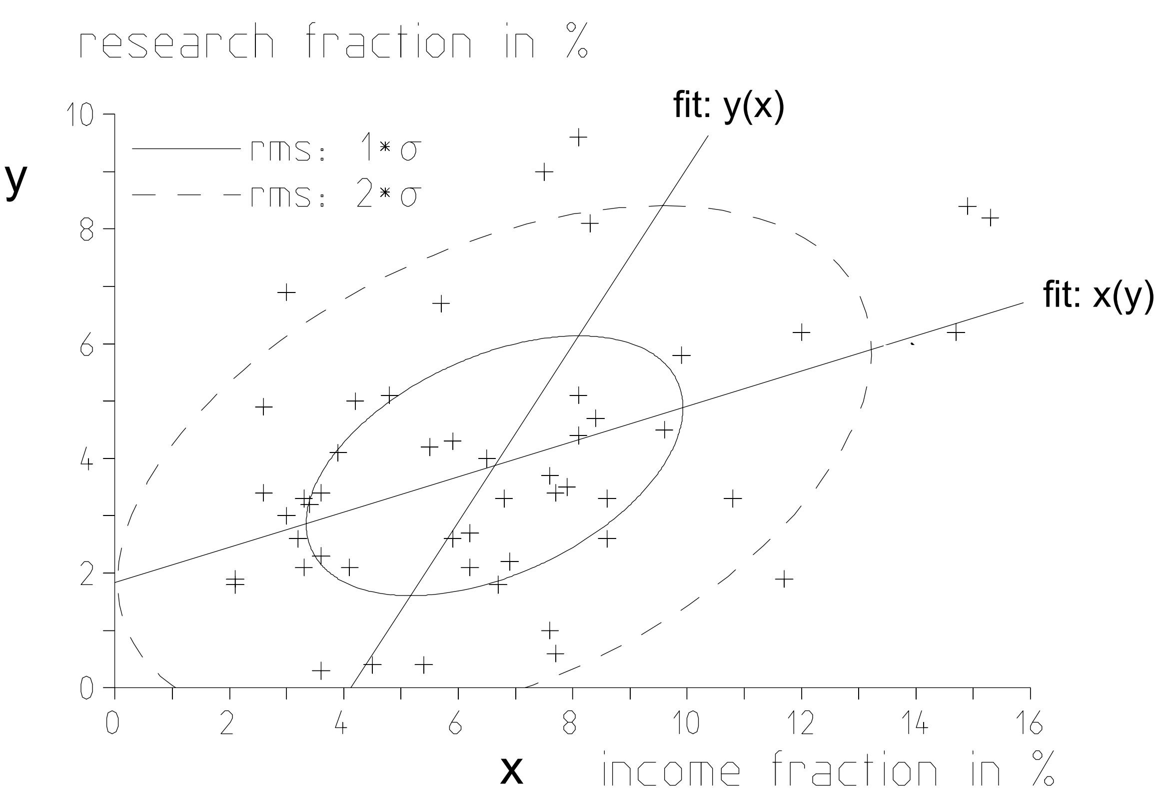
(from physics today, march and september 1978)

x = income / sales

y = research budget / sales

There are 3 possibilities to show a correlation:

1. linear fit of y(x) :   income stimulates research !
2. linear fit of x(y) :   research stimulates income !
3. correlation ellipse from <x y> : high income ⇔ strong research



exponential growth with compound interest

For a quick estimate of exponential growth one can use

$$e^7 \approx 2^{10} \approx 10^3$$

example:

With an interest rate of p(%) it takes T_2 years

to double an initial investment.

$$T_2 = 70 \text{ years/p(\%)}$$

(70=100 ln2)

To have an increase by a factor of 1‘000

it takes T_{1000} years

$$T_{1000} = 10\ T_2 = 700 \text{ years/p(\%)}$$